Random Vibration of Cylindrical Shells

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Nomenclature

C_0 = power spectral density function of the white noise
E = modulus of elasticity
$G_{ff}(\omega)$ = power spectral density function of the input excitation
h,L,R = thickness, length, and radius of the shell, respectively
x = distance measured along the axis of the shell
δ_{jk} = Kronecker delta
ω_j = circular frequency (rad/s) for the jth mode
ν = Poisson's ratio
ρ = mass density
ξ_j = damping ratio corresponding to mode j

Introduction

THIN shell-type structural elements are commonly employed in aerospace systems. It is often essential that such structures be designed for severe environments of random forcing fields. Recently, Elishakoff et al.[1] have presented an explicit solution for the stationary response of simply supported thin cylindrical shells subjected to a concentrated radial ring load which is random in time domain. In their investigation, the influence of modal cross-corrections on the mean-square velocity of the shell is found to be significantly considerable.

The present work is concerned with the stationary and transient random vibration of cylindrical shells. The method by Madsen and Krenk[2] is used in the analysis. This formulation provides mean-square response of the shell through the use of simple expressions for the covariance functions and is convenient in regard to the computational aspects. The stationary part of the present results is in very good agreement with those presented in Ref. 1.

Method of Analysis

In accordance with the shell theory by Flügge,[3] the axisymmetric radial vibration of a thin elastic circular cylinder can be described by the following equation:

$$\frac{w}{R^2}+\left(\frac{h^2}{12R^4}\right)\left(\frac{R^4\partial^4 w}{\partial x^4}\right)+\left[\frac{\rho h(1-\nu^2)}{E}\right]\frac{\partial^2 w}{\partial t^2}=0 \quad (1)$$

If the shell is simply suported at its ends $x=0$ and $x=L$, the equation for the dimensionless frequency becomes:

$$\Omega_j^2=\frac{\rho L^2\omega_j^2}{E}=\left(\frac{L}{R}\right)^2+\left(\frac{h}{R}\right)^2\frac{(j\pi)^2}{12(1-\nu^2)} \quad (2)$$

corresponding to the mode shape

$$\Phi_j(x)=\sqrt{2}\,\sin(j\pi x/L) \quad (3)$$

which satisfies the following orthogonality condition:

$$\left(\frac{1}{L}\right)\int_0^L \Phi_j(x)\Phi_k(x)\mathrm{d}x=\delta_{jk} \quad (4)$$

Consider a thin elastic shell subjected to the axisymmetric ring load at $x=a$. The time-history of the load is assumed to be a wide band stationary random process. Using the normal mode approach in the frequency domain, the mean-square response of the shell can be determined by the following double summation[1]:

$$E\langle w^2(x)\rangle=\sum_{j=i}^{\infty}\sum_{k=i}^{\infty}\Phi_j(a)\Phi_j(x)\Phi_k(a)\Phi_k(x)R_{jk} \quad (5)$$

R_{jk} represents

$$R_{jk}(t_1,t_2)=\int_{-\infty}^{\infty}G_{ff}(\omega)H_j(\omega,t_1)\bar{H}_k(\omega,t_2)\times\exp\{i\omega(t_1-t_2)\}\mathrm{d}\omega \quad (6)$$

$H_j(\omega,t)$ may be referred to as the time-dependent transfer function.

$$H_j(\omega,t)=H_j(\omega)[1-\{g_j(t)+i\omega h_j(t)\}\exp(-i\omega t)]$$

$$H_j(\omega)=1/(\omega_j^2-\omega^2+i2\xi_j\omega_j\omega)$$

$$g_j(t)=\exp(-\omega_j\xi_j t)\{\cos(\omega_{dj}t)+c_j\sin(\omega_{dj}t)\},\quad t\geq 0$$
$$=0,\quad t<0$$

$$h_j(t)=\exp(-\omega_j\xi_j t)(1/\omega_{dj})\sin(\omega_{dj}t),\quad t\geq 0$$
$$=0,\quad t<0$$

$$\omega_{dj}=\omega_j\sqrt{(1-\xi_j^2)},\quad c_j=\xi_j/\sqrt{(1-\xi_j^2)} \quad (7)$$

and the overbar represents the complex conjugate functions. Earlier studies[2,4] have shown that sufficiently accurate stationary and transient mean-square response of a multidegree of freedom system subjected to a random forcing field described by the power spectral density function $G_{ff}(\omega)$ can be calculated by approximating the actual excitation to an equivalent white noise. Accordingly, Eq. (6) can be reduced to the following simplified form:

$$R_{jk}(t_1,t_2)=\pi C_0[\alpha_{jk}\{g_j(t_1-t_2)-g_j(t_1)g_k(t_2)+\gamma_{jk}h_j(t_1)h_k(t_2)\}+\beta_{jk}\{h_j(t_1-t_2)+h_j(t_1)g_k(t_2)-h_k(t_2)g_j(t_1)\}] \quad (8)$$

where symbols α_{jk}, β_{jk}, and γ_{jk} are

$$\alpha_{jk}=4(\omega_j\xi_j+\omega_k\xi_k)/D$$

$$\beta_{jk}=(\omega_k\omega_k-\omega_j\omega_j)/D$$

$$\gamma_{jk}=\omega_j\omega_k(\omega_j\xi_k+\omega_k\xi_j)/(\omega_j\xi_j+\omega_k\xi_k)$$

$$D=(\omega_j\omega_j-\omega_k\omega_k)^2+4\omega_j\omega_k(\omega_j\xi_k+\omega_k\xi_j)(\omega_j\xi_j+\omega_k\xi_k) \quad (9)$$

The covariance function to compute the mean-square values of the transient displacement is obtained by letting $t_1=t_2=t$ in Eq. (8). Therefore

$$R_{jk}(t)=\pi C_0[\alpha_{jk}\{1-g_j(t)g_k(t)+\gamma_{jk}h_j(t)h_k(t)\}+\beta_{jk}\{h_j(t)g_k(t)-h_k(t)g_j(t)\}] \quad (10)$$

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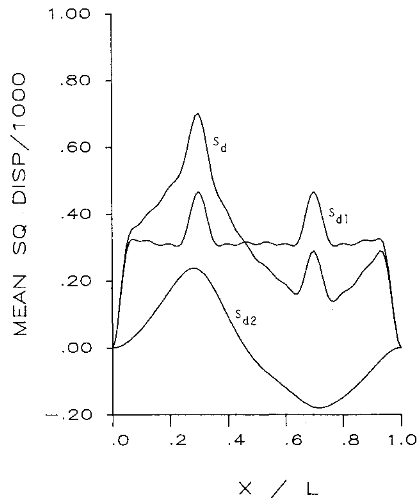

Fig. 1 Stationary mean-square displacement, $S_d = E\langle w^2\rangle [\rho h^2 L/\sqrt{(E/\rho)}/C_0]$ vs X/L for shell with $R/L = 0.5$, $h/L = 0.01$, $B = 0.01$, and $\Omega_c = 2\pi$.

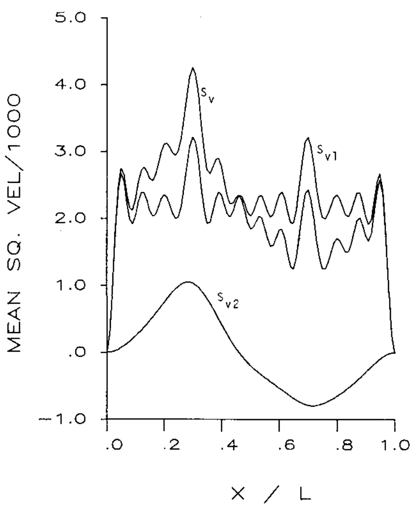

Fig. 2 Stationary mean-square velocity, $S_v = E\langle \dot{w}^2\rangle [\rho h^2 L/\sqrt{(E/\rho)}/C_0]$ vs X/L for shell with $R/L = 0.5$, $h/L = 0.01$, $B = 0.01$, and $\Omega_c = 2\pi$.

In a similar manner, the mean square transient velocity can be determined by differentiating Eq. (8) with respect to t_1 and t_2 and then equating $t_1 = t_2 = t$ as discussed in Refs. 4 and 5. This results in the following expression:

$$\dot{R}_{jk}(t) = \pi C_0 [\omega_j^2 \alpha_{jk} \{1 - \omega_k^2 h_j(t) h_k(t)\}$$
$$- \gamma_{jk} f_j(t) f_k(t) + \omega_j \beta_{jk} \{2\xi_j + \omega_j h_j(t) f_k(t)$$
$$- (\omega_k/\omega_j)\omega_k h_k(t) f_j(t)\}]$$
$$f_j(t) = \exp(-\omega_j \xi_j t)\{\cos(\omega_{dj} t) - c_j \sin(\omega_{dj} t)\} \tag{11}$$

At infinitely large t_1 and t_2 having finite $t_1 - t_2$, it is found that the transient response approaches stationarity for which Eqs. (10) and (11) reduce to the following simple expressions:

$$r_{jk}(0) = \pi C_0 \alpha_{jk} \text{ and } \dot{r}_{jk}(0) = \pi C_0 (\omega_j^2 \alpha_{jk} + 2\omega_j \xi_j \beta_{jk}) \tag{12}$$

Results and Discussions

The analytical method discussed above is applied to the cylindrical shell with the following parameters: $R/L = 0.5$, $h/L = 0.01$, and $a/L = 0.3$. Numerical computations are performed for the excitation band $\Omega_c = 2\pi$ within which there are 14 natural modes of the shell. Equation (2) of the present analysis yields higher values of natural frequencies than those obtained from a similar expression in Ref. 1. The difference ranges from approximately 5% in the lowest natural mode of 0.5% in the 14th mode.

The mean-square responses of the following dimensionless forms are calculated from Eq. (12):

$$S_d = E\langle w^2\rangle \{\rho^2 h^2 L\sqrt{(E/\rho)}/Co\}' = S_{d_1} + S_{d_2} \tag{13}$$
$$S_v = E\langle \dot{w}^2\rangle \{\rho^2 h^2 L/\sqrt{(E/\rho)}/Co\} = S_{v_1} + S_{v_2} \tag{14}$$

In Eqs. (13) and (14) subscripts d and v denote displacement and velocity of the shell, respectively. Similarly, subscripts 1 and 2 correspond to the response due to autocorrelation and cross-correlation terms, respectively. Stationary responses are calculated for the dimensionless damping parameter $B = \xi L\sqrt{(\rho/E)} = 0.01$ for all modes. Figures 1 and 2 show the distribution of mean-square displacement and velocity along the axis of the shell. The auto-correlation sums S_{d_1} and S_{v_1} are symmetric about the midpoint, i.e., $x/L = 0.5$. The cross-correlation sums S_{d_2} and S_{v_2} are asymmetric with reference to the axis of the shell. The influence of S_{d_2} and S_{v_2} on the overall response of the shell is evidently considerable and cannot be omitted from the calculation. The distribution pattern and numerical values of the mean-square velocity shown in Fig. 2 are compared with the results of

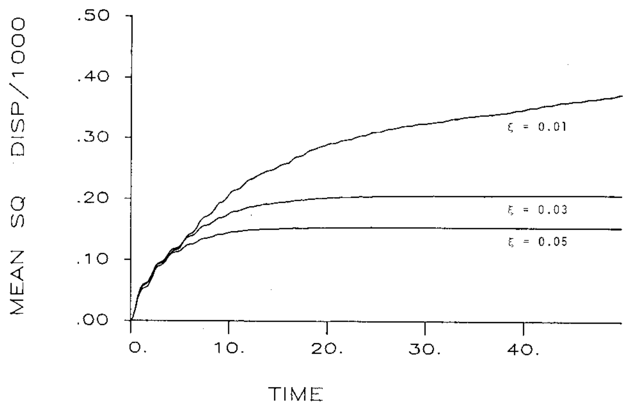

Fig. 3 Transient mean-square displacement, $S_d = E\langle w^2\rangle [\rho h^2 L/\sqrt{(E/\rho)}/C_0]$ vs time $= t/[R\sqrt{(\rho/E)}]$ for shell with $R/L = 0.5$, $h/L = 0.01$, $a/L = 0.3$, and $\Omega_c = 2\pi$.

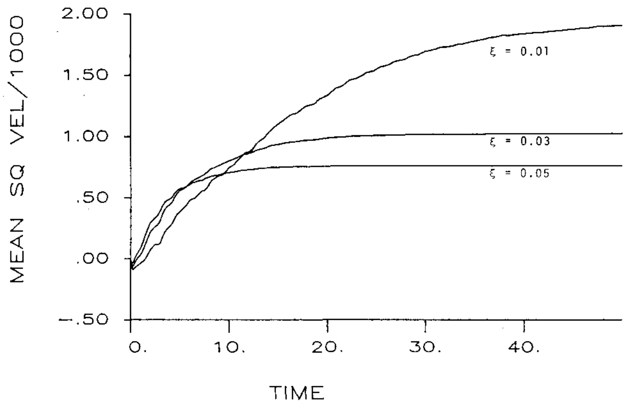

Fig. 4 Stationary mean-square velocity, $S_v = E\langle \dot{w}^2\rangle [\rho h^2 L/\sqrt{(E/\rho)}/C_0]$ vs X/L for shell with $R/L = 0.5$, $h/L = 0.01$, $B = 0.01$, and $\Omega_c = 2\pi$.

Ref. 1. The agreement is very good with only 0.5% difference in the peak value of S_v at $x/L=0.3$.

The mean-square transient responses at $x=a$ have been calculated using Eqs. (10) and (11). The damping ratios $\xi=0.01$, 0.03, and 0.05 are assumed to be constant for modes 1–14. The transient responses are initially zero and then approach their stationary values for large time instant as shown in Figs. 3 and 4. It is seen from Eqs. (10–12) that the present method brings about a direct relationship between the stationary and transient response of the shell.

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